一般化された分配則としてのGrayテンソル積

Generalization of formal monad theory to lax functors

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Overview of my thesis

1. Motivation (monad <=> lax functor)

2. Recall: formal theory of monads (Section 3)

    A. Monads vs. Adjunctions
    B. Distributive laws

3. Generalization to lax functors

    A. Lax doctrinal adjunctions (Section 4)
    The contents in this section was an existing result.
    B. Generalized distributive laws (Section 5)

Lax functor b/w bicategories

A **lax functor** $F\colon \mathscr{C} \rightsquigarrow \mathscr{D}$ between bicategories $\mathscr{C}$, $\mathscr{D}$ is a 2-categorically weakened notion of functors,

- which maps all the 0,1,2-cells in $\mathscr{C}$ to those in $\mathscr{D}$,
- but which only preserves horizontal compositions up to comparison maps.

That is, there are following 2-cells

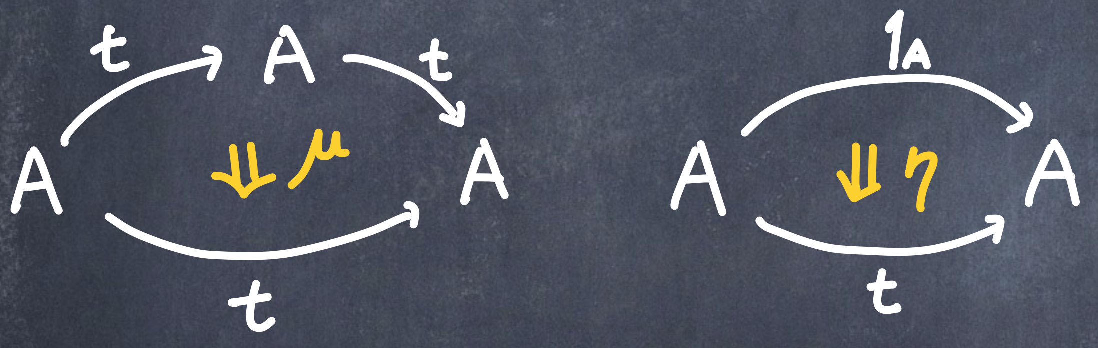
$$\begin{tikzcd}
& FB \arrow[dr, "Fg"] & \\
FA \arrow[ur, "Ff"] \arrow[rr, bend right, "F(gf)"'] \arrow[rr, phantom, "\Downarrow F^2_{f,g}"] & & FC
\end{tikzcd}
\qquad
\begin{tikzcd}
FC \arrow[r, bend left, "1_{FC}"] \arrow[r, bend right, "F1_C"'] \arrow[r, phantom, "\Downarrow F^0_C"] & FC
\end{tikzcd}$$

satisfying some coherence conditions.

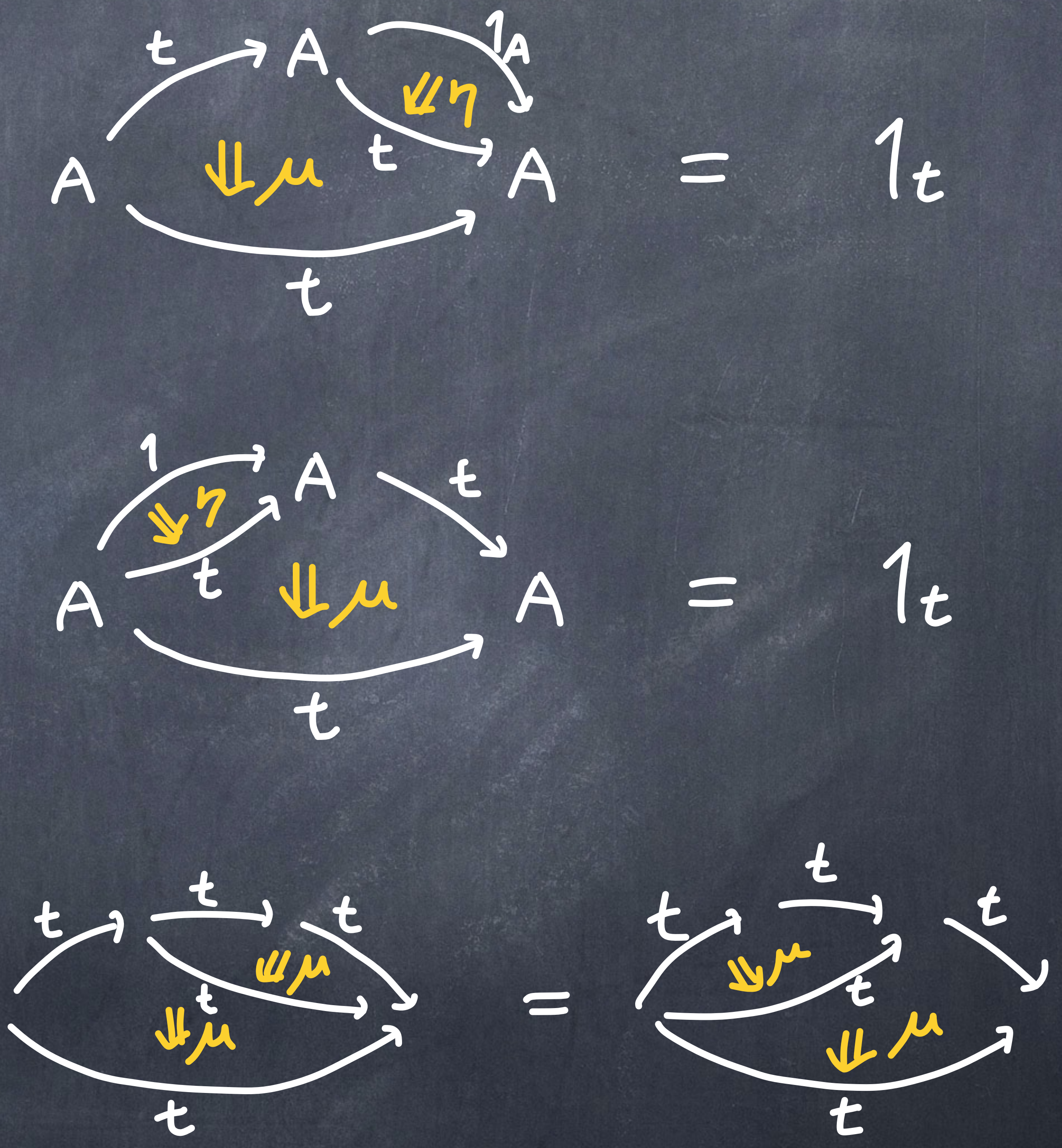
Monads as lax functors

A monad in a bicategory $\mathscr{D}$ consists of:

- An endo-1-cell $t\colon A \to A$
- multiplication 2-cell $\mu\colon tt \Rightarrow t$
- unit 2-cell $\eta\colon 1_A \Rightarrow t$



satisfying three axioms on the right.



The data above is equivalent to a lax functor $\bar{t}\colon \mathbf{1} \rightsquigarrow \mathscr{D}$

Lax functors as generalized monads

- A lax functor $\bar{t} \colon \mathbf{1} \rightsquigarrow \mathscr{D}$ is a monad.
- Let $\mathscr{M}$ be a monoidal category, and assume it as a single-object bicategory $\Sigma\mathscr{M}$. Then, a lax functor $\Sigma\mathscr{M} \rightsquigarrow \mathscr{D}$ is a graded monad.
- Let $\mathbb{B}$, $\mathbb{C}$ be categories and $\mathrm{End}(\mathbb{C})$ be the monoidal category of endo-functors. Then, a category graded monad is a lax functor $\mathbb{B} \rightsquigarrow \Sigma\mathrm{End}(\mathbb{C})$

Question

To what extent do lax functors behaves like monads?

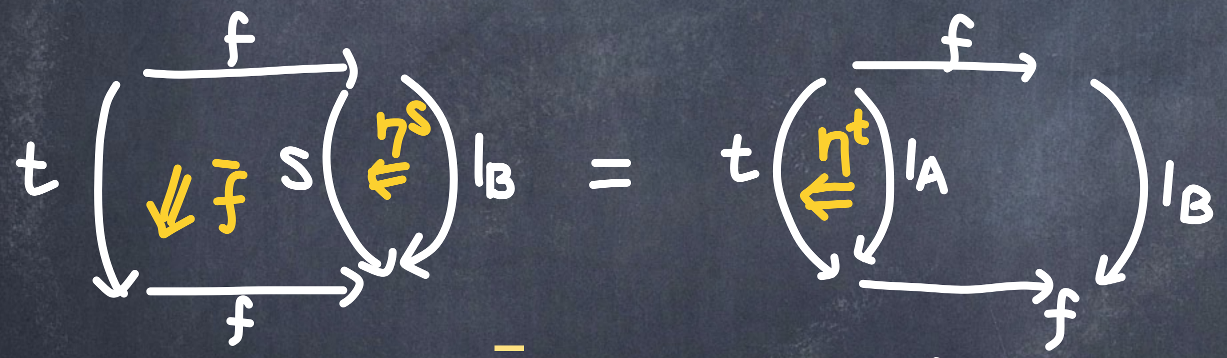
Key idea: Rephrase "monad" as "lax functor $\mathbf{1} \rightsquigarrow \mathscr{D}$".
(But not trivial)

monad (1): 2-category of monads

A lax monad morphism $(f, \bar{f})$ from $t\colon A \to A$ to $s\colon B \to B$ consists of a 1-cell $f\colon A \to B$ and a 2-cell,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & B \arrow[d, "s"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

which is compatible to μ and η.



A 2-cell $\alpha\colon (f, \bar{f}) \Rightarrow (g, \bar{g})$ between lax monad morphisms is defined as a 2-cell $\alpha\colon f \Rightarrow g$ which is compatible to $\bar{f}$ and $\bar{g}$.

The 2-category of monads in $\mathscr{K}$ will be denoted by $\mathrm{mnd}_l(\mathscr{K})$.

generalization (1): 2-category of lax functors

Notation. The 2-category of w-functors and w'-transformations will be denoted by ${}^{w}_{w'}[\mathscr{A}, \mathscr{K}]$.

e.g.) ${}^{2}_{L}[\mathscr{A}, \mathscr{K}]$ = 2-functor + lax transformation + modification.

The augmented simplex category Δ_a has natural numbers $\{0,1,2,\dots\}$ as objects and monotone maps as morphisms.

$$\begin{tikzcd}
0 \arrow[r] & 1 \arrow[r, shift left=2] \arrow[r, shift right=2] & 2 \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & 3 \arrow[l, shift right=6] \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[l, shift left=6] & \cdots
\end{tikzcd}$$

The free 2-category of monad $\mathbf{Mnd}$ is a single-object 2-category whose hom-category is Δ_a.

Prop. Let $\mathscr{K}$ be a 2-category. The 2-category of monads in $\mathscr{K}$ can be presented as,

$$\mathrm{mnd}_l(\mathscr{K}) \;\cong\; {}^{L}_{L}[\mathbf{1}, \mathscr{K}] \;\cong\; {}^{2}_{L}[\mathbf{Mnd}, \mathscr{K}].$$

Generally⋯

Thm. For each bicategory $\mathscr{C}$, there is a 2-category $\overline{\mathscr{C}}$ called the lax functor classifier with the universal property:

$$ {}^{L}_{w}[\mathscr{C}, \mathscr{K}] \;\cong\; {}^{2}_{w}[\overline{\mathscr{C}}, \mathscr{K}] $$

$\left(\overline{\mathbb{1}} \cong \mathbb{M}\mathrm{nd} \right)$

monad (2): relation to adjunctions

An adjunction $f \dashv g$ in $\mathscr{C}$ consists of:

- 1-cells $f\colon A \to B$ and $g\colon B \to A$
- Together with 2-cells ε and η

$$\begin{tikzcd}
A \arrow[rr, bend left, "1_A"] \arrow[dr, "f"'] \arrow[rr, phantom, "\Downarrow\eta"] & & A \\
& B \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dr, "f"] & \\
B \arrow[ur, "g"] \arrow[rr, bend right, "1_B"'] \arrow[rr, phantom, "\Downarrow\varepsilon"] & & B
\end{tikzcd}$$

satisfying the triangular identities

$$\begin{tikzcd}
A \arrow[rr, equal] \arrow[dr, "f"'] \arrow[r, phantom, "\Downarrow\eta"'] & & A \arrow[dr, "f"] & \\
& B \arrow[ur, "g"] \arrow[rr, equal] \arrow[r, phantom, "\Downarrow\varepsilon"] & & B
\end{tikzcd}
= 1_f \quad ,$$

$$\begin{tikzcd}
& A \arrow[rr, equal] \arrow[dr, "f"] \arrow[r, phantom, "\Downarrow\eta"'] & & A \\
B \arrow[ur, "g"] \arrow[rr, equal] \arrow[r, phantom, "\Downarrow\varepsilon"] & & B \arrow[ur, "g"'] &
\end{tikzcd}
= 1_g$$

Prop. Any adjunction $f \dashv g$ induces a monad gf.

$$\text{unit} = \begin{tikzcd}
\cdot \arrow[rr, bend left] \arrow[dr, "f"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \cdot \\
& \cdot \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\text{multiplication} = \begin{tikzcd}
\cdot \arrow[r, "f"'] & \cdot \arrow[r, bend left, "g"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "\Downarrow\varepsilon"] & \cdot \arrow[r, bend left, "f"] & \cdot \arrow[r, "g"'] & \cdot
\end{tikzcd}$$

Prop. If a 2-category $\mathscr{A}$ has sufficient limits/colimits, any monads in $\mathscr{A}$ arose from Eilenberg-Moore/Kleisli adjunctions.

Prop. Let the 2-category of right adjoints $\mathrm{radj}(\mathscr{K})$ be the full sub-2-category of $\mathscr{K}^{\to}$.
If a 2-category $\mathscr{A}$ admits all Eilenberg-Moore objects, then $\mathrm{mnd}_l(\mathscr{K})$ is a reflective sub-2-category of $\mathrm{radj}(\mathscr{K})$.

generalization (2): relation to lax doctrinal adjunctions

(This is the contents of Section 4 in my thesis, which turned out that it was already known in Steve Lack's paper "Morita contexts as lax functors".)

There is a 2-comonad whose 2-category of lax coalgebras and lax morphisms is ${}^{L}_{L}[\mathscr{A},\mathscr{K}]$. With some 2-monad theoretical discussion, I found the following, which was first found in Ross Street's paper "Two constructions on lax functors".

Let us consider the following data: (pointwise adjunction)

- A 2-functor $\overline{F}\colon \mathscr{A} \to \mathscr{K}$,
- A family of adjunctions $FA \overset{\longrightarrow}{\underset{\longleftarrow}{\perp}} \overline{F}A$ ($A \in \mathscr{A}$).

The relation between the data above and a lax functor is the same as the relation between an adjunction and a monad. That is···

generalization (2): relation to lax doctrinal adjunctions

Thm. Any pointwise adjunction $FA \mathrel{\substack{\xrightarrow{l_A} \\ \perp \\ \xleftarrow[r_A]{}}} \overline{F}A$ induces a lax functor $F\colon \mathscr{A} \to \mathscr{K}$ defined by

	$\mathcal{A}$		$\mathcal{K}$
object	A	$\longmapsto$	FA
1-cell	$f\colon A \to B$	$\longmapsto$	$FA \xrightarrow{l_A} \bar{F}A \xrightarrow{\bar{F}f} \bar{F}B \xrightarrow{r_B} FB$
2-cell	$\alpha : f \Rightarrow B$	$\longmapsto$	$r_B \,.\, \bar{F}\alpha \,.\, l_A$

comparison 2-cell

$$F^2_{f,g} :$$

$$\begin{tikzcd}
& & & {} \arrow[dr, "l_B"] & & & \\
{} \arrow[r, "l_A"] & {} \arrow[r, "\bar{F}f"] & {} \arrow[ur, "r_B"] \arrow[rr, equal, bend right] & \Downarrow \varepsilon_B & {} \arrow[r, "\bar{F}g"] & {} \arrow[r, "r_C"] & {}
\end{tikzcd}$$

$$F^0_A :$$

$$\begin{tikzcd}
FA \arrow[rr, equal, bend left] \arrow[dr, "l_A"'] & \Downarrow \eta_A & FA \\
& \bar{F}A \arrow[ur, "r_A"'] &
\end{tikzcd}$$

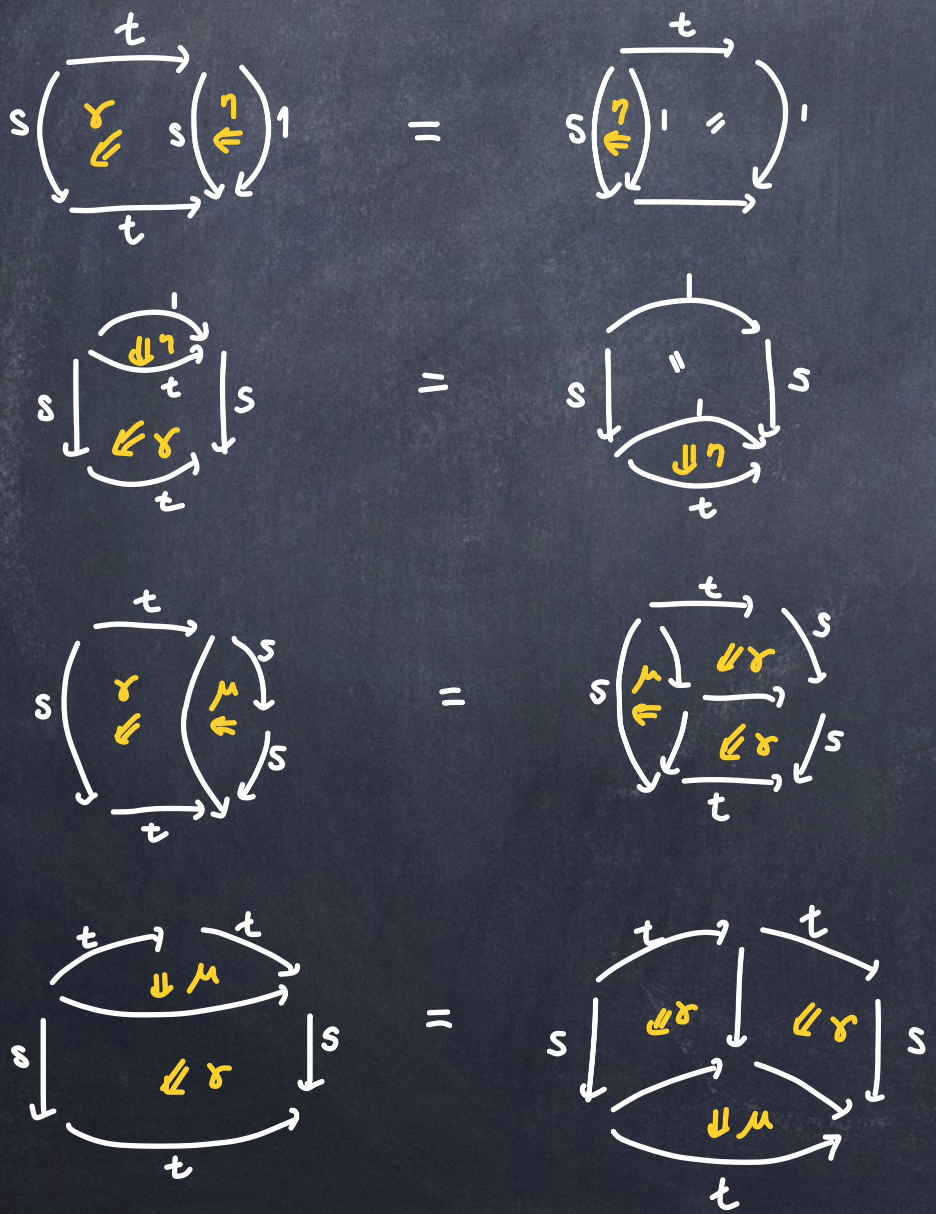
Thm. Let $\mathscr{K}$ be a complete 2-category, and $\mathscr{A}$ be a small 2-category. Given any lax functor $F\colon \mathscr{A} \rightsquigarrow \mathscr{K}$, there is a 2-functor $\mathscr{R}F\colon \mathscr{A} \to \mathscr{K}$ called the Eilenberg-Moore 2-functor and pointwise adjunction $FA \mathrel{\substack{\longrightarrow \\ \perp \\ \longleftarrow}} \mathscr{R}FA$, which induces F.

Thm. ${}^{L}_{L}[\mathscr{A}, \mathscr{K}]$ is a reflective sub-2-category of pointwise adjunctions.

monad (3): distributive law

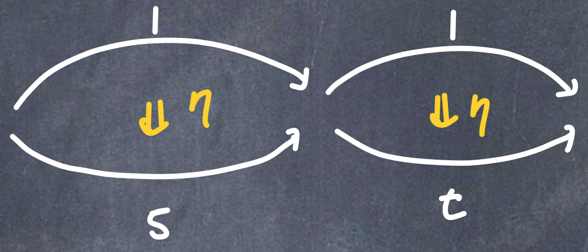
Let s, t be monads on A in $\mathscr{K}$.

An distributive law of t over s is a 2-cell $\gamma \colon st \Rightarrow ts$ satisfying:

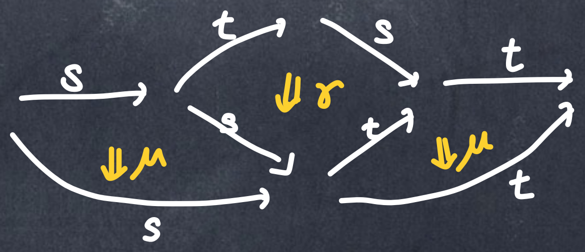


Prop. If $\gamma \colon st \Rightarrow ts$ is a distributive law, there is a monad structure on ts called the composition of t and s defined by:

Unit



Multiplication



monad (3): distributive law

Thm. [Beck69] Let t and s be monads on the same object A in $\mathscr{K}$. All the following give the same data.

1. A distributive law $\gamma \colon st \Rightarrow ts$.

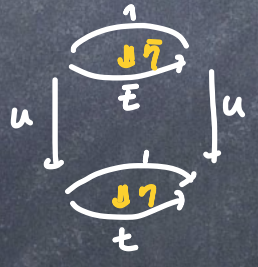
2. A monad in the 2-category of monads $\mathrm{mnd}_l(\mathscr{K})$.

3. A lifting $\bar{t}$ of the monad t to the Eilenberg-Moore object A^s.

monad $(\bar{t}, \bar{\eta}, \bar{\mu})$ on A^s s.t.

$$\begin{tikzcd}
A^s \arrow[r, "\bar{t}"] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowright"] & A^s \arrow[d, "u"] \\
A \arrow[r, "t"'] & A
\end{tikzcd}$$

,



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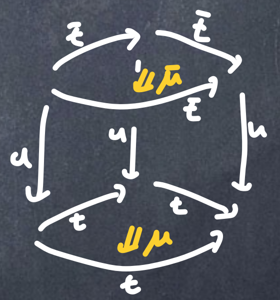
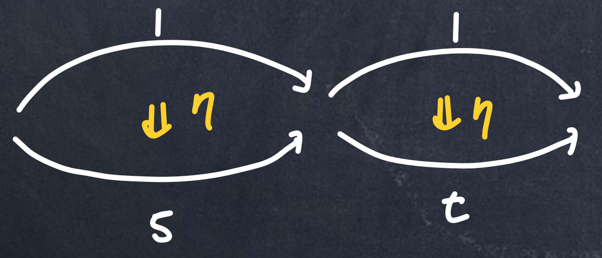
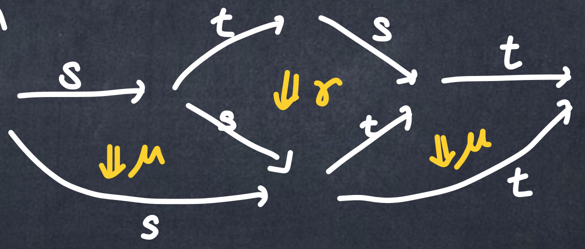


4. A 2-cell $\gamma \colon st \Rightarrow ts$ which makes ts a monad in the following way.

Unit



Multiplication



generalization (3): generalized distributive law

Thm. [Gray74] There is a non-symmetric monoidal structure on the category of 2-categories 2-Cat$_0$ called lax Gray tensor $\otimes_l$ product with left and right closed structure,

$$\mathscr{A} \otimes_l (\,-\,) \quad \dashv \quad {}^{2}_{L}[\mathscr{A}, -\,]$$

$$(\,-\,) \otimes_l \mathscr{B} \quad \dashv \quad {}^{2}_{Op}[\mathscr{B}, -\,].$$

A distributive law in $\mathscr{K}$ was a monad in $\mathrm{mnd}_l(\mathscr{K})$. So the 2-category of distributive laws is defined by

$${}^{2}_{L}\left[\mathbf{Mnd},\, {}^{2}_{L}\left[\mathbf{Mnd},\, \mathscr{K}\right]\right] \;\cong\; {}^{2}_{L}\left[\mathbf{Mnd} \otimes_l \mathbf{Mnd},\, \mathscr{K}\right]$$

Which is also isomorphic to

$${}^{2}_{L}\left[\bar{\mathbf{1}},\; {}^{2}_{L}\left[\bar{\mathbf{1}},\, \mathscr{K}\right]\right] \;\cong\; {}^{2}_{L}\left[\bar{\mathbf{1}} \otimes_l \bar{\mathbf{1}},\, \mathscr{K}\right]$$

generalization (3): generalized distributive law

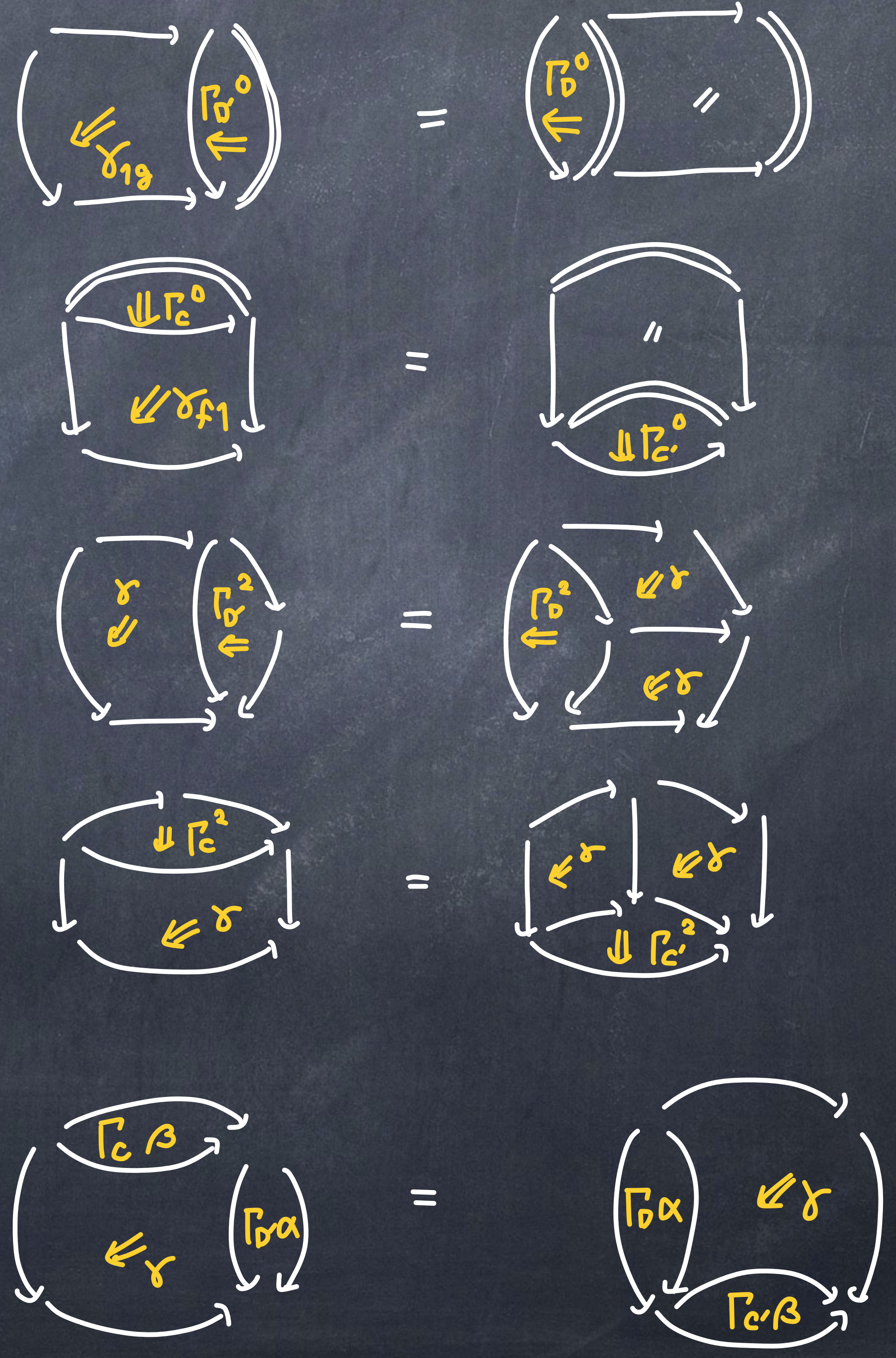
Def. [Nikolić19] A generalized distributive law in $\mathscr{K}$ is a 2-functor from $\overline{\mathscr{C}} \otimes_l \overline{\mathscr{D}}$ to $\mathscr{K}$, which is equivalent to a lax functor $\mathscr{D} \rightsquigarrow {}^{L}_{L}[\mathscr{C}, \mathscr{K}]$

This is also equivalent to the following data:

- Lax functors $\Gamma(C, -): \mathscr{D} \rightsquigarrow \mathscr{K}$
- Lax functors $\Gamma(-, D): \mathscr{C} \rightsquigarrow \mathscr{K}$
- 2-cells for each pair of 1-cells ($f: C \to C',\ \ g: D \to D'$)

$$\begin{tikzcd}
\Gamma_{CD} \arrow[r, "\Gamma_{Cg}"] \arrow[d, "\Gamma_{fD'}"'] \arrow[dr, phantom, "\Swarrow \gamma_{f,g}"] & \Gamma_{CD'} \arrow[d, "\Gamma_{fD'}"] \\
\Gamma_{C'D} \arrow[r, "\Gamma_{C'g}"'] & \Gamma_{C'D'}
\end{tikzcd}$$

satisfying



Lax functor classifier and lax Gray tensor

Prop. Sending $\mathscr{A}$ to the 2-functor ${}^{2}_{L}\left[\mathscr{A}, -\right]$ defines a (strict) monoidal functor $(2\text{-Cat}_0,\ \otimes_l,\ \mathbf{1}) \to \left({}^{2}_{2}[2\text{-Cat},\ 2\text{-Cat}],\ \circ,\ \mathrm{Id}\right)$.

Thm. There is a following lax/oplax monoidal adjunction between $(\mathbf{BiCat}_0^{\mathrm{Lax}},\ \times,\ \mathbf{1})$ and $(2\text{-}\mathbf{Cat}_0,\ \otimes_l,\ \mathbf{1})$.

$$\begin{tikzcd}
2\text{-Cat}_0 \arrow[r, bend right, "\text{lax}", "\text{inclusion}"'] \arrow[r, phantom, "\perp"] & \mathrm{BiCat}_0^{\mathrm{Lax}} \arrow[l, bend right, "\text{oplax}", "\overline{\mathscr{E}} \mapsfrom \mathscr{E}"']
\end{tikzcd}$$

Cor. For each bicategory $\mathscr{C}$, its lax functor classifier $\overline{\mathscr{C}}$ has a canonical lax Gray comonoid structure, which induce a 2-monad ${}^{2}_{L}\left[\overline{\mathscr{C}}, -\right]$ on 2-Cat.

2-monad ${}^{2}_{L}\left[\overline{\mathscr{C}}, -\right]$

When $\mathscr{C} = \mathbf{1}$, the 2-monad ${}^{2}_{L}\left[\overline{\mathbf{1}}, -\right]$ is isomorphic to $\mathrm{mnd}_l(\mathscr{K})$.

And the multiplication is defined by the composition of monads.

$$\mathrm{mnd}_l(\mathrm{mnd}_l(\mathscr{K})) \;\to\; \mathrm{mnd}_l(\mathscr{K})$$

distributive law ↦ composed monad

The multiplication of the 2-monad ${}^{2}_{L}\left[\overline{\mathscr{C}}, -\right]$ can be seen as a generalization of composition of monads.

$${}^{L}_{L}\left[\mathscr{C}, {}^{L}_{L}\left[\mathscr{C}, \mathscr{K}\right]\right] \;\to\; {}^{L}_{L}\left[\mathscr{C}, \mathscr{K}\right]$$

generalized distributive law ↦ “composed” lax functor

Eilenberg-Moore and distributive law

Let $\mathscr{A}$, $\mathscr{B}$ be 2-categories and $\Gamma\colon \overline{\mathscr{A}} \otimes_l \overline{\mathscr{B}} \to \mathscr{K}$ be a generalized distributive law. Suppose $\Gamma'\colon \mathscr{B} \rightsquigarrow {}^{L}_{L}[\mathscr{A}, \mathscr{K}]$ is the corresponding lax functor.

Thm. Let All the following defines isomorphic 2-functors from $\mathscr{A} \times \mathscr{B}$.

1. The Eilenberg-Moore 2-functor of the composite $\mathscr{B} \overset{\Gamma'}{\rightsquigarrow} {}^{L}_{L}[\mathscr{A}, \mathscr{K}] \xrightarrow{EM = \mathscr{R}} {}^{2}_{2}[\mathscr{A}, \mathscr{K}]$.
2. The composite $\mathscr{B} \xrightarrow{\mathscr{R}\Gamma'} {}^{L}_{L}[\mathscr{A}, \mathscr{K}] \xrightarrow{EM} {}^{2}_{2}[\mathscr{A}, \mathscr{K}]$.
3. The Eilenberg-Moore 2-functor of the composite $\mathscr{A} \times \mathscr{B} \overset{J}{\rightsquigarrow} \overline{\mathscr{A}} \otimes_l \overline{\mathscr{B}} \xrightarrow{\Gamma} \mathscr{K}$.

J: canonical lax functor

$A \times B \rightsquigarrow \bar{A} \otimes_l \overline{B}$

Conclusion

I studied how formal monad theory can be generalized to lax functors, and

- I redescovered the results in Section 8 of Steve Lack's paper "Morita contexts as lax functors".
- I proveded some statements related to generalized distributive laws:
  - A lax/oplax monoidal adjunction between $(\mathbf{BiCat}_0^{\mathrm{Lax}}, \times, \mathbf{1})$ and $(2\text{-}\mathbf{Cat}_0, \otimes_l, \mathbf{1})$, and found a canonical comonoid structure on $\overline{\mathscr{C}}$.
  - Relation to Eilengerg-Moore construction of lax functors.