Programming with Quantum-Controlled Quantum Channels

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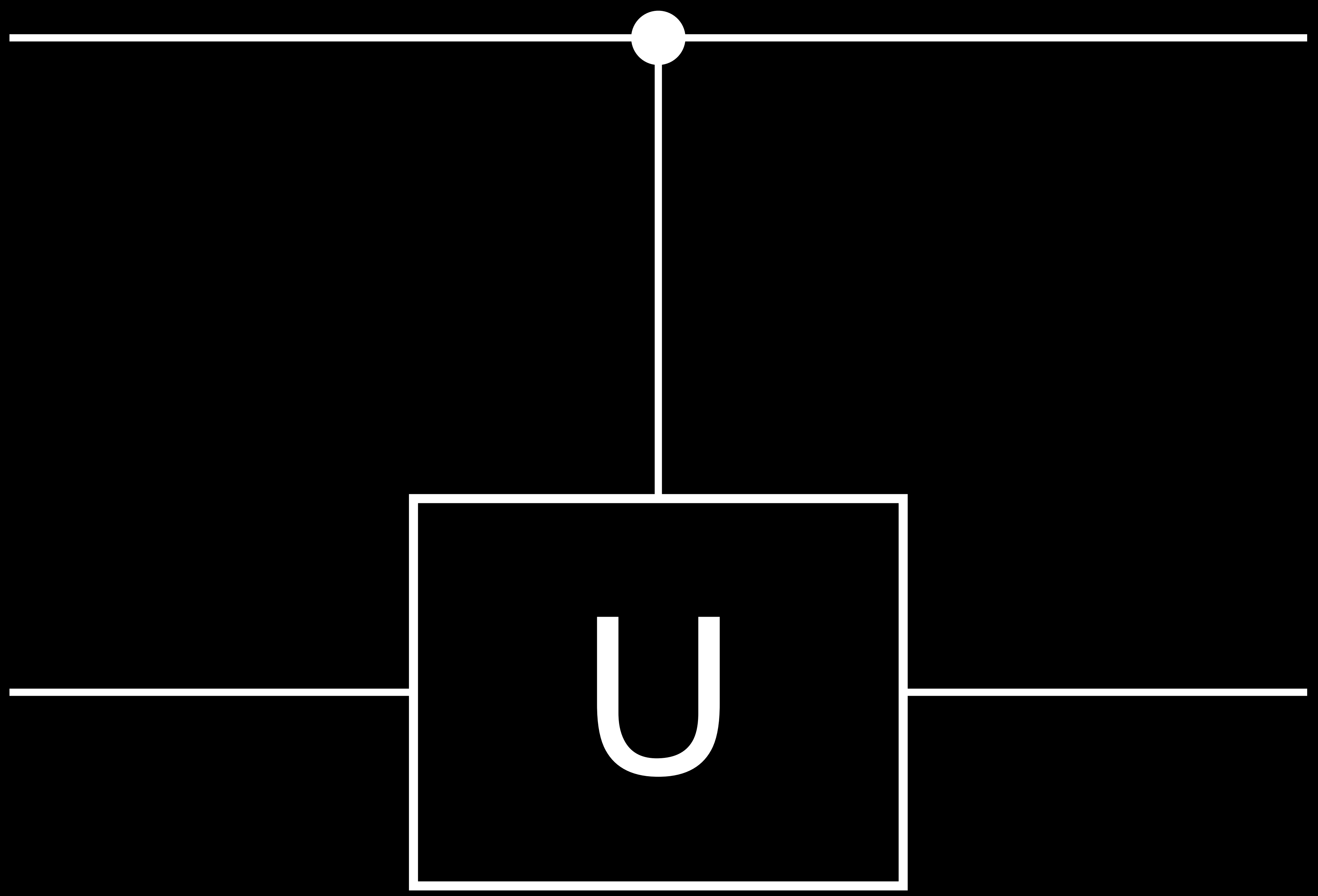
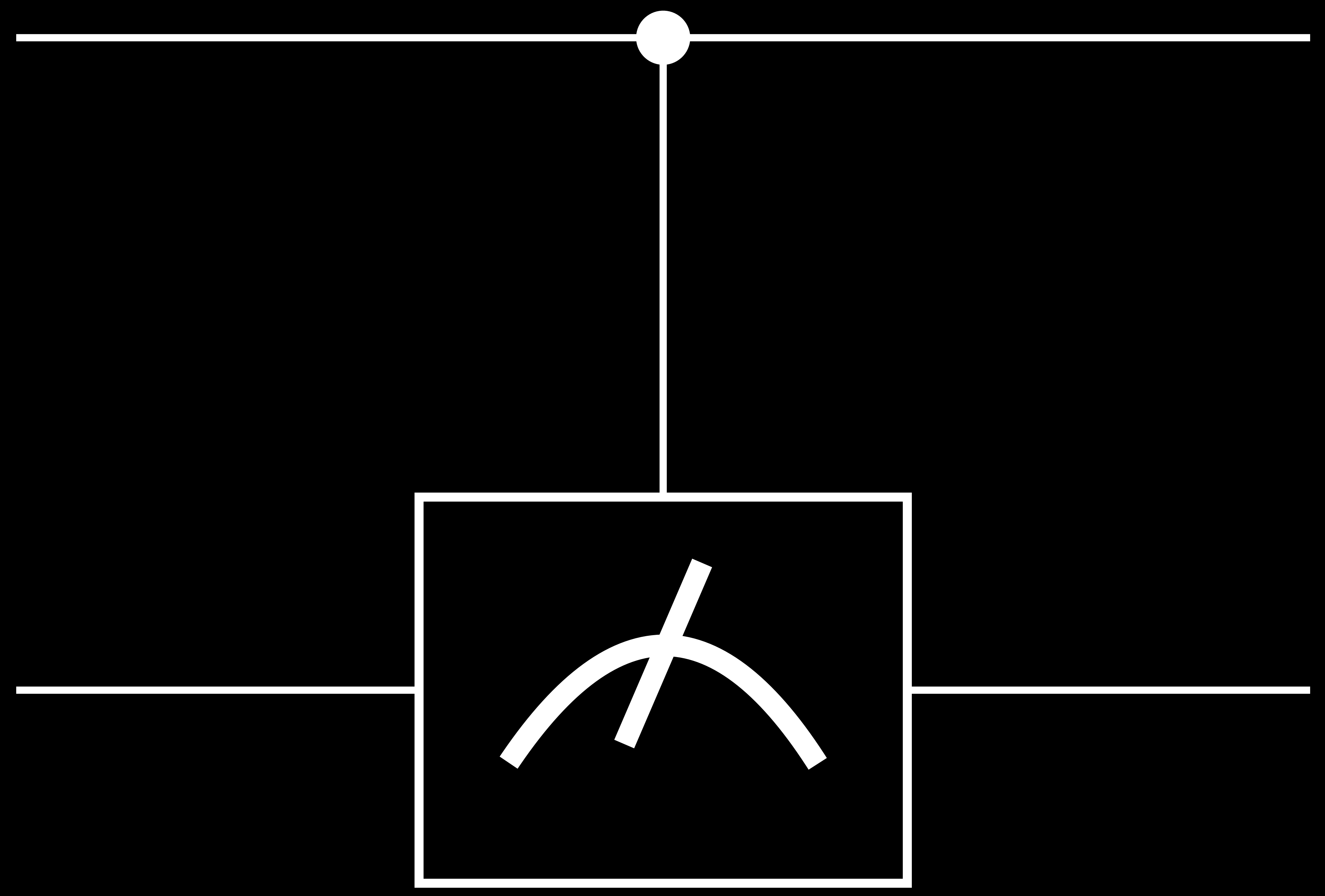
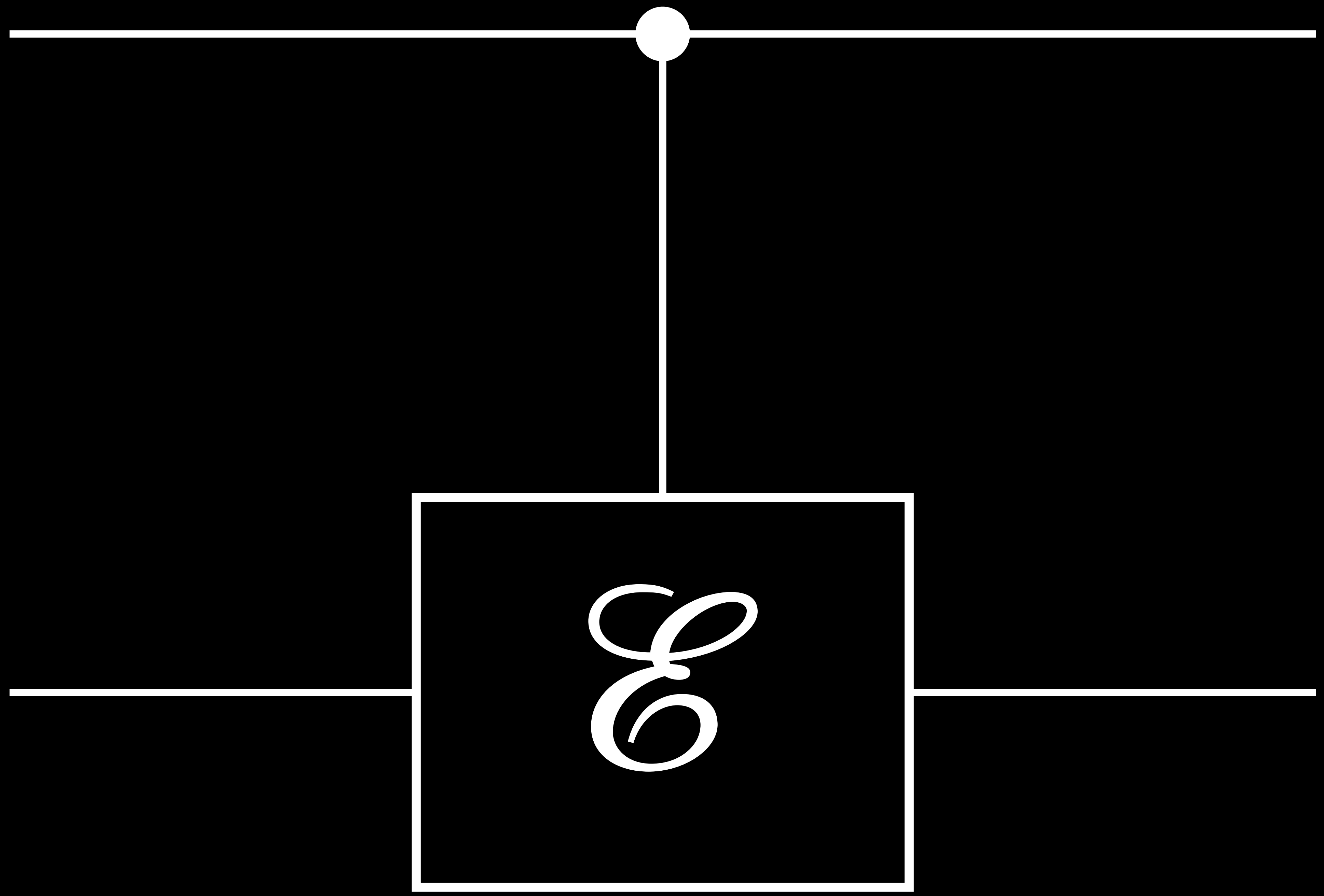


U: unitary

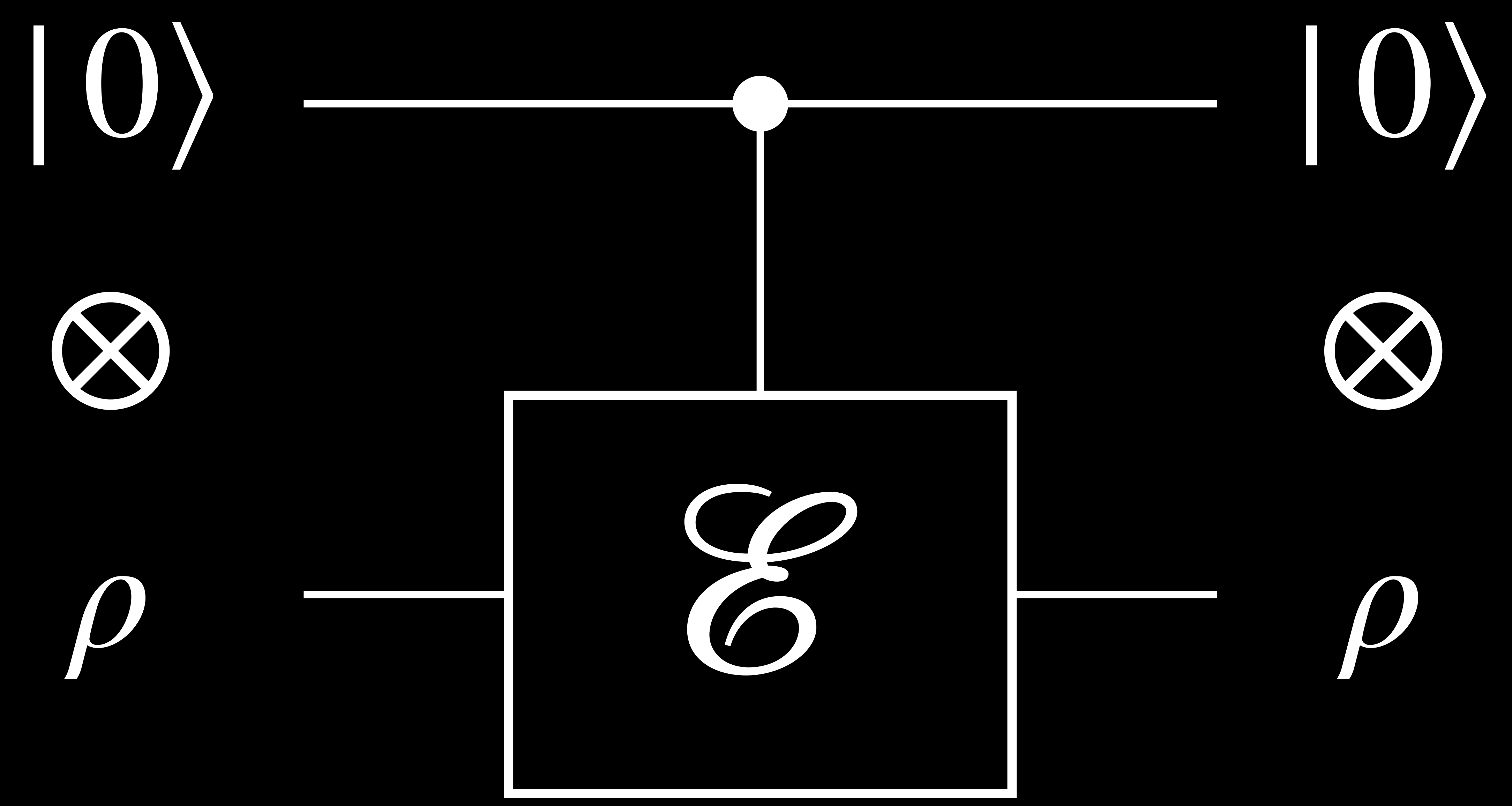




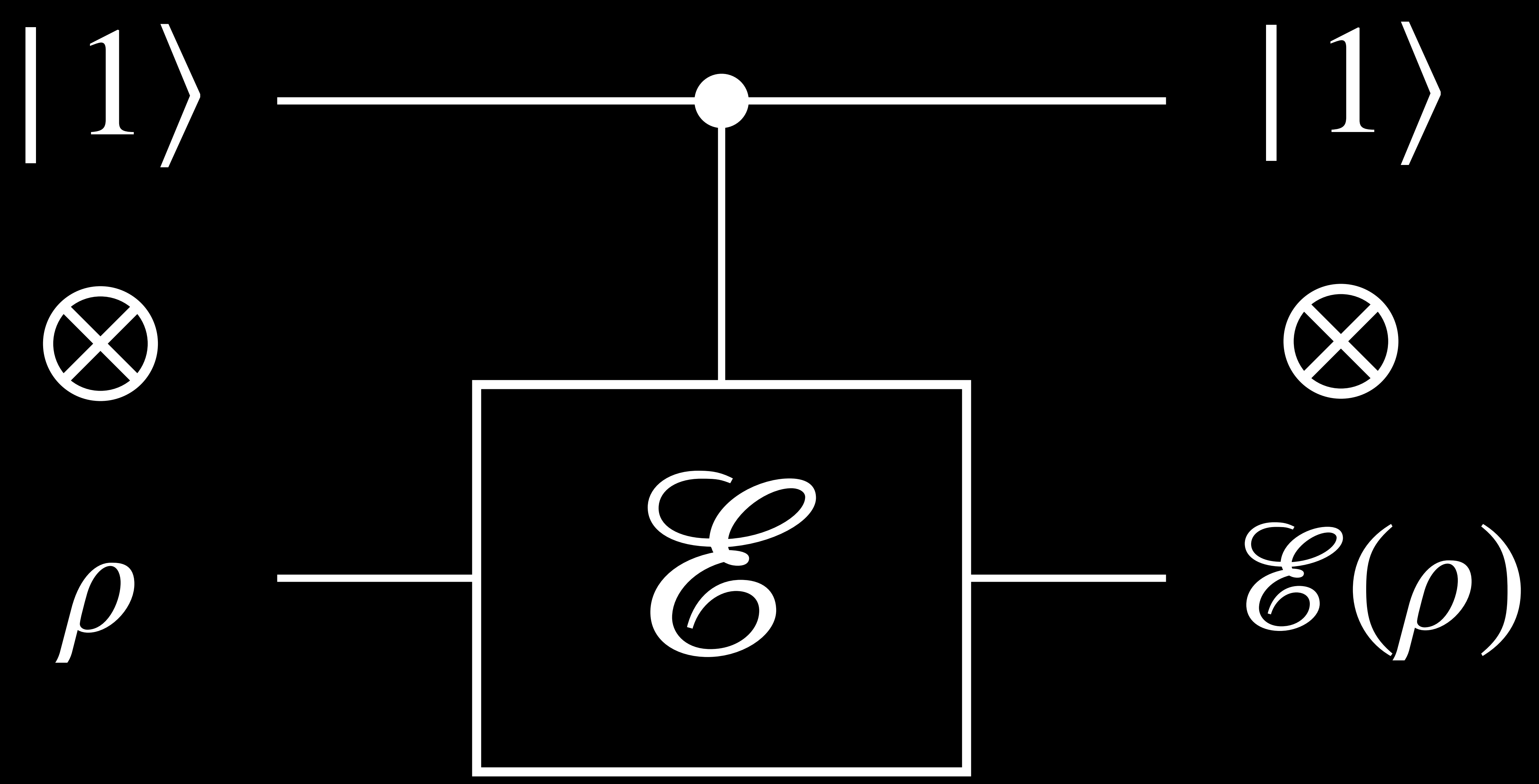
$\mathcal{E}$: channel

Axioms for controlled-$\mathscr{E}$

Axiom 1.

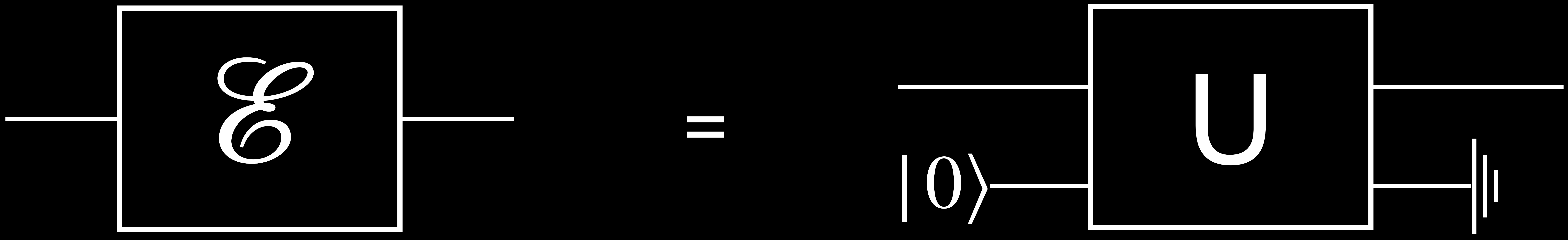


Axiom 2.



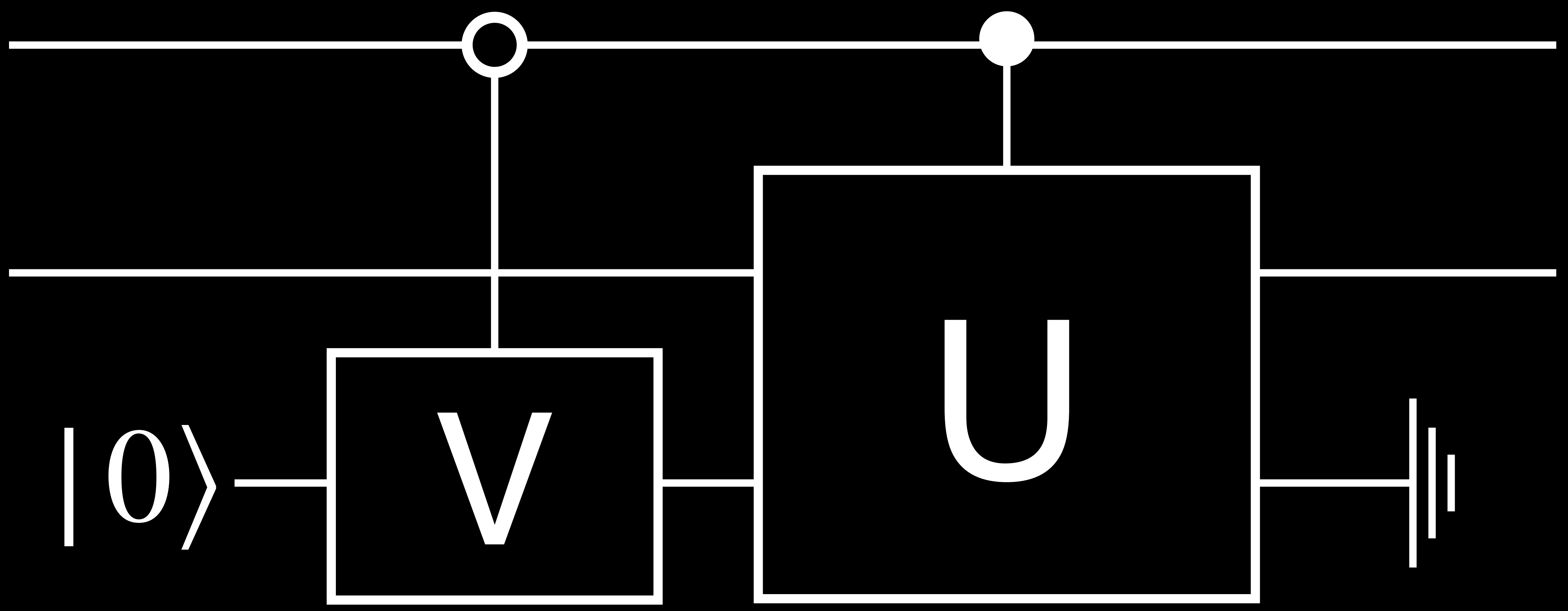
Controlled-$\mathscr{E}$ by Purification

Take *any* Stinespring dilation

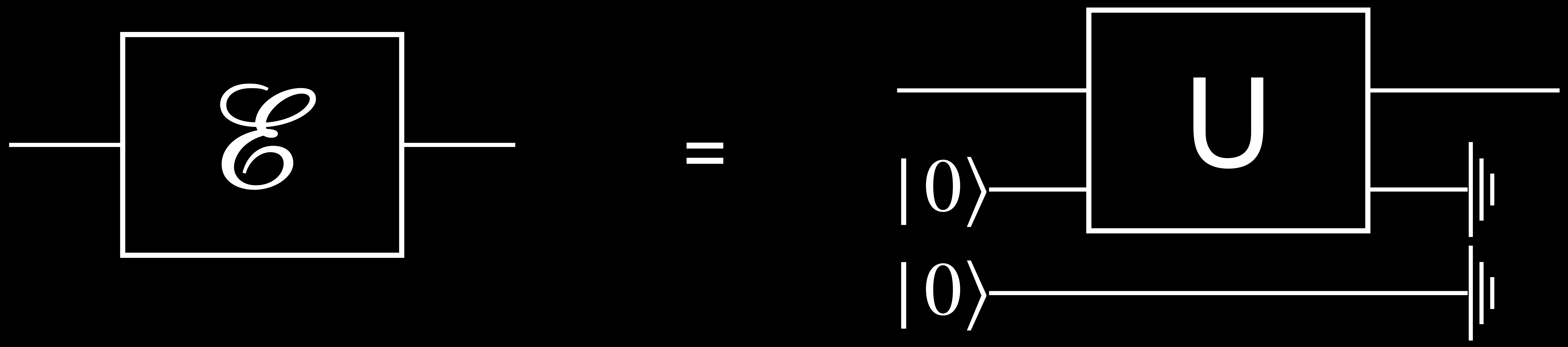


Then

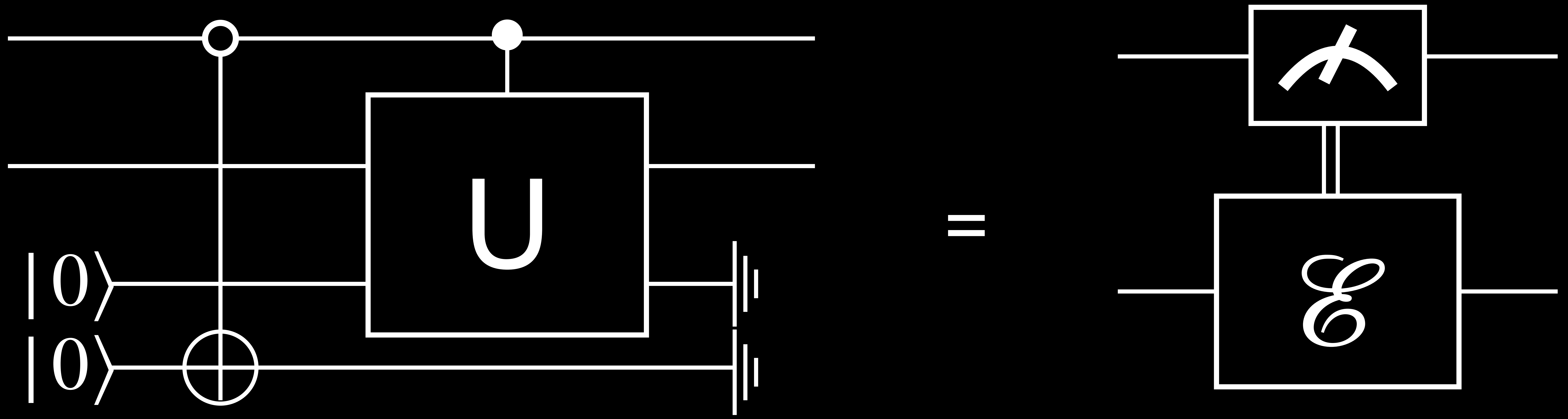
∀ V: unitary,



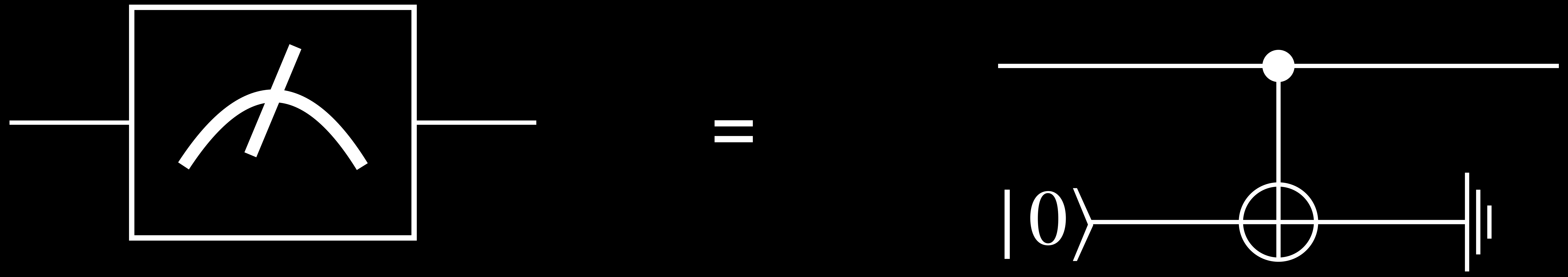
Example of Purification 1.



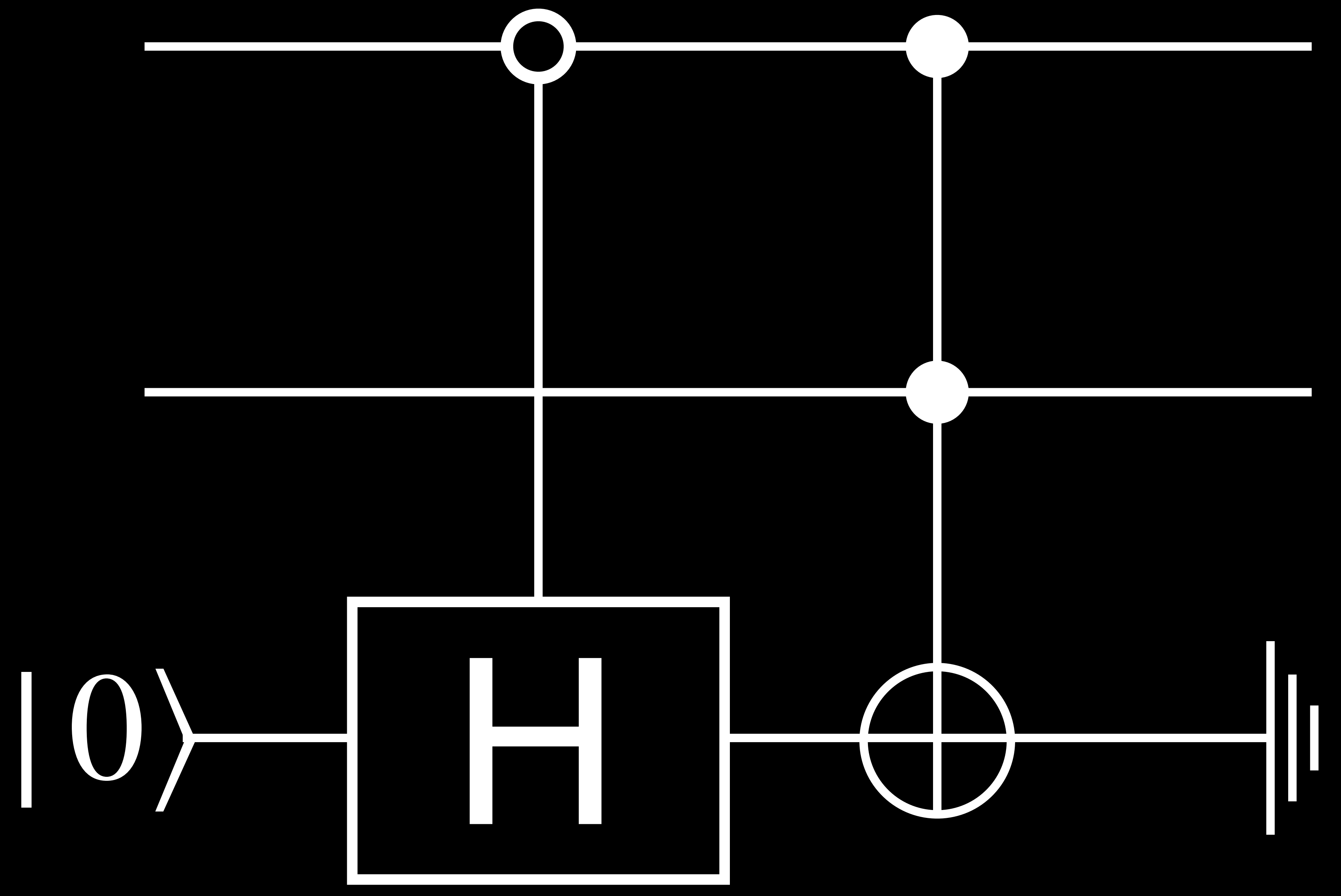
Then



Example of Purification 2.



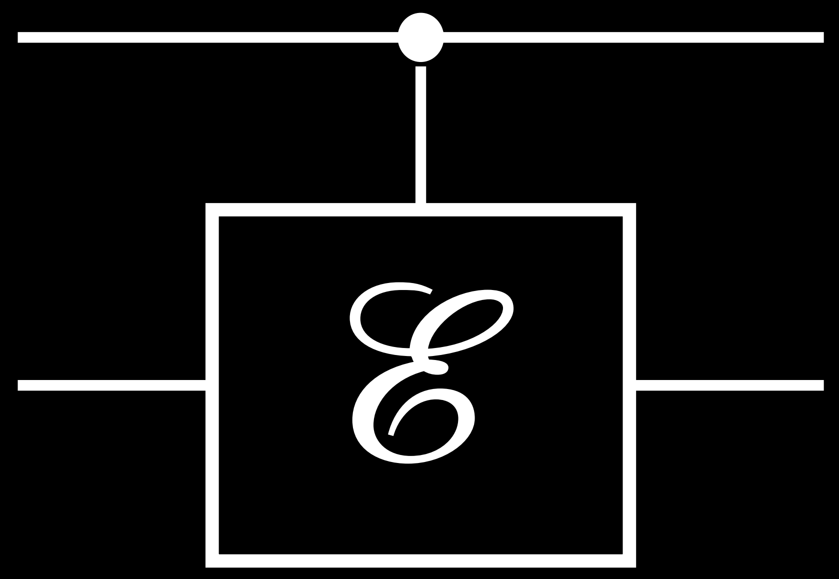
Then



Cannot tell the state of the control qubit

No Go Theorem

For each $\mathscr{E}$,



is determined only up to some unitary.

$\simeq$ There is no unique canonical choice

Ill-definedness Issue

Second Story: Quantum SWITCH

For each $\mathscr{E}, \mathscr{F} : (\mathbf{C}^2)^{\otimes n} \longrightarrow (\mathbf{C}^2)^{\otimes n}$,

take their Kraus decompositions

$$\{K_i\}_{i \in I} \qquad \{L_j\}_{j \in J}.$$

SWITCH($\mathscr{E}, \mathscr{F}$) :=

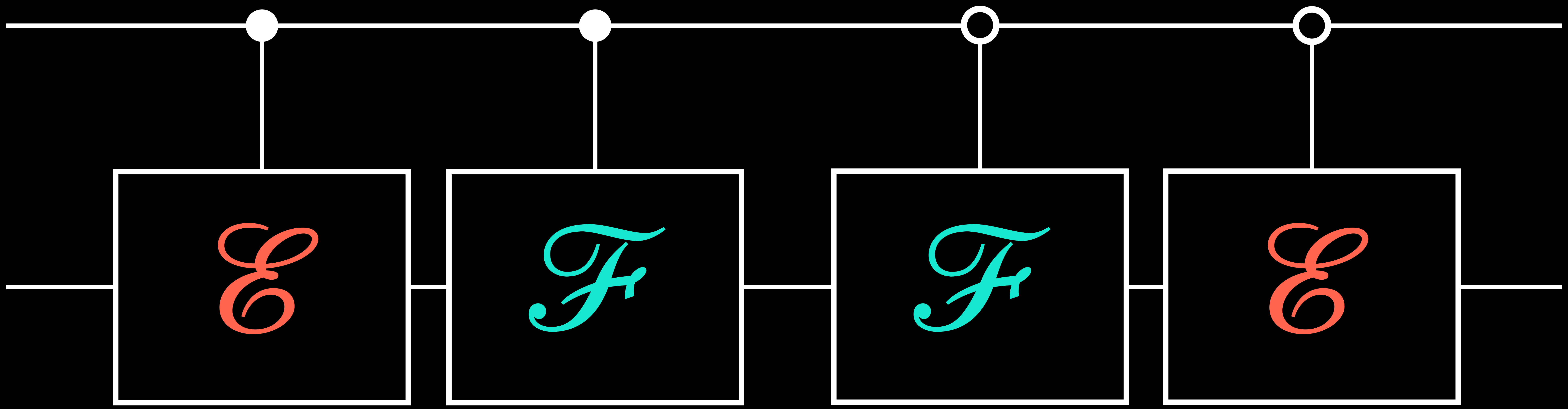
$$\{ \; |0\rangle\langle 0| \otimes K_i \circ L_j + |1\rangle\langle 1| \otimes L_j \circ K_i \; \}_{i,j \in I \times J}$$

Roughly speaking,

(c: control qubit)

SWITCH($\mathscr{E}$, $\mathscr{F}$) $\simeq$

if c **then** $\mathscr{F} \circ \mathscr{E}$ **else** $\mathscr{E} \circ \mathscr{F}$



Well-defined

Second Story: Quantum SWITCH

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Question: What's the difference?

- General Controlled-$\mathscr{E}$

$$\textbf{if } c \textbf{ then } \mathscr{E}$$

Ill-definedness issue

- Depend on decomposition

- Quantum SWITCH

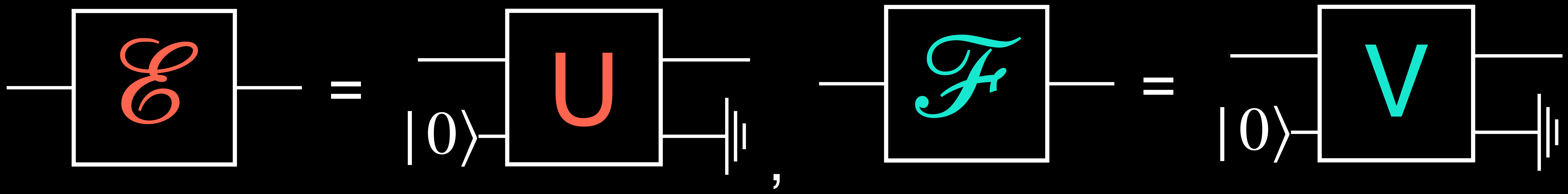
$$\textbf{if } c \textbf{ then } \mathscr{F} \circ \mathscr{E} \quad \textbf{else } \mathscr{E} \circ \mathscr{F}$$

Well-defined map

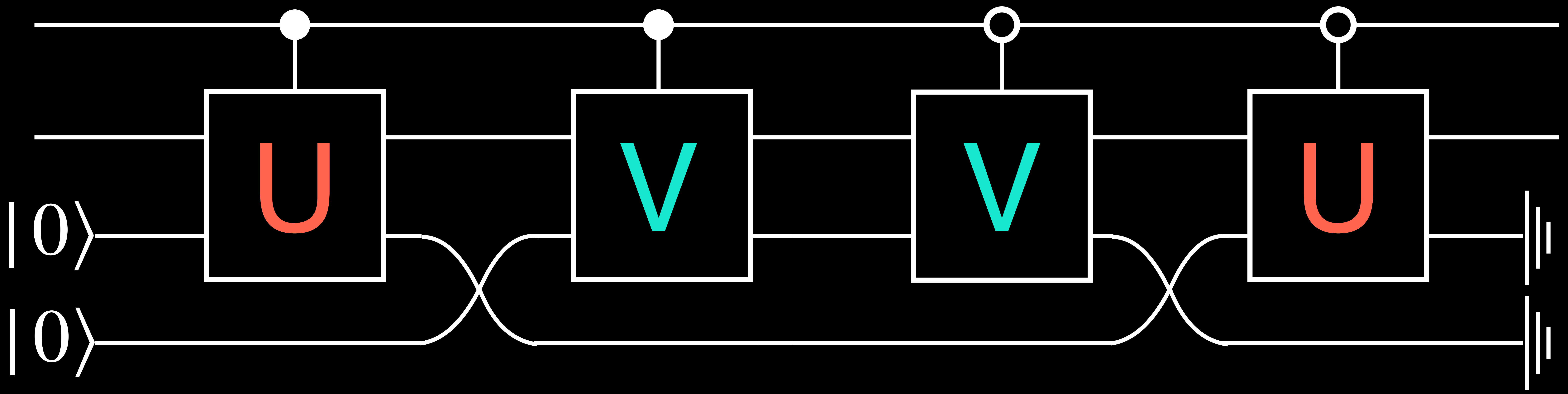
- Independent of choices of Kraus-decomp.

Q. What is the difference?

Analysing SWITCH

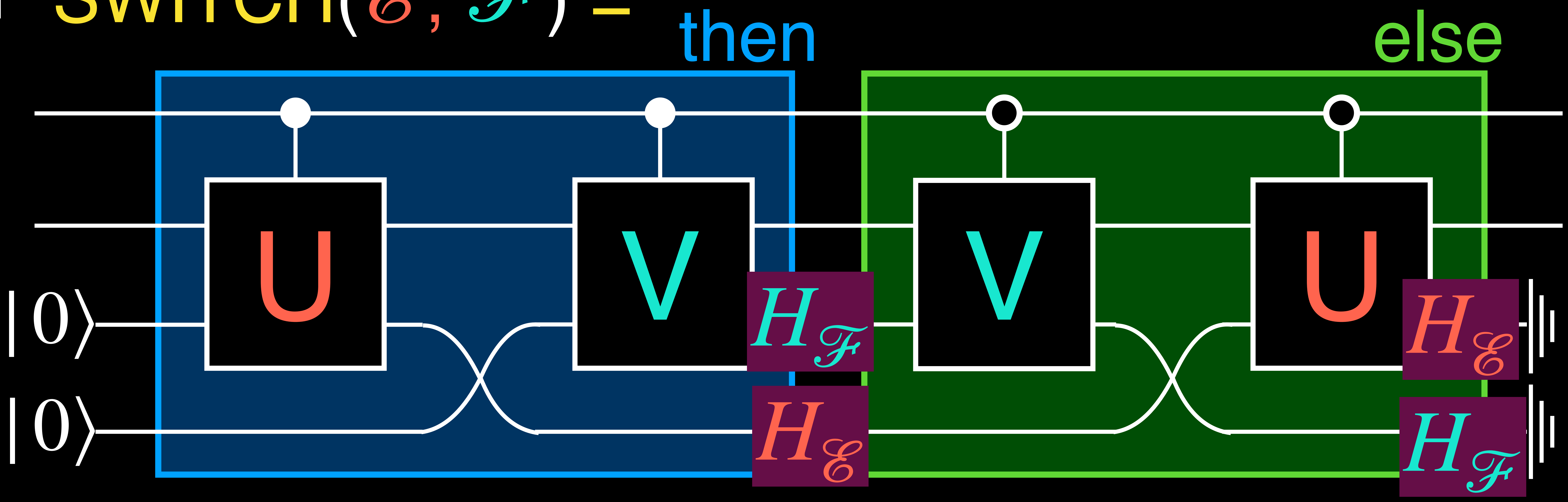


Then SWITCH($\mathcal{E}$, $\mathcal{F}$) =



Analysing SWITCH

Then SWITCH($\mathscr{E}$, $\mathscr{F}$) =



Observation: correspondence problem

- General Controlled-$\mathscr{E}$

  if c **then** $\mathscr{E}$
  else **id**

  No correspondence between then/else

- Quantum SWITCH

  if c **then** $\mathscr{F} \circ \mathscr{E}$
  else $\mathscr{E} \circ \mathscr{F}$

  Then/else clause share a common auxiliary space

Goal: Programming Language with quantum control & Type System

Every program should have a canonical semantics

i.e., ✗ General control

✓ SWITCH

Idea: Linear Types for qif

Correspondence of auxiliary spaces
between then/else $H_{\mathscr{E}}$ $H_{\mathscr{F}}$

Idea: Linear Types for qif

Correspondence of auxiliary spaces

Correspondence of channel usage

$$\textbf{qif}\ \ c\ \ \textbf{then}\ \ \mathscr{F} \circ \mathscr{E}$$
$$\textbf{else}\ \ \mathscr{E} \circ \mathscr{F}$$

Idea: Linear Types for qif

Correspondence of auxiliary spaces

Correspondence of channel usage

Idea: Linear usage of channels

= linear types

qif c **then** $\mathscr{F} \circ \mathscr{E}$
else $\mathscr{E} \circ \mathscr{F}$

Rough Idea:

Γ = channels used in branches

$$\frac{\Gamma \vdash M_1 : A \qquad \Gamma \vdash M_2 : A}{\Gamma, x : \text{qbit} \vdash \text{qif } x \text{ then } M_1 \text{ else } M_2 : \text{qbit} \otimes A}$$

Idea: Linear Types for qif

Correspondence of auxiliary spaces

Correspondence of channel usage

Idea: Linear usage of channels

= *linear types*

$$\textbf{qif}\ c\ \textbf{then}\ \mathscr{F} \circ \mathscr{E}\ \textbf{else}\ \mathscr{E} \circ \mathscr{F}$$

Rough Idea:

{ f, g } = channels used in branches

$$\frac{f, g \vdash M_1 : A \qquad f, g \vdash M_2 : A}{f, g, x\text{: qbit} \vdash \text{qif } x \text{ then } M_1 \text{ else } M_2 : \text{qbit} \otimes A}$$

Language with 2 derivations

Types ::= qbit | bool | $A \otimes B$ | $A \multimap B$

Terms ::= (linear lambda calculus)

| U | $|0\rangle$ | meas

| qif M then N_1 else N_2 | $\cdots$

Type derivations

$$\Gamma \vdash_{\text{Hilb}} M : A \qquad \Gamma \vdash_{\text{CPM}} M : A$$

Sublanguages Hilb & CPM

$\Gamma \vdash_{\text{Hilb}} \mathrm{M} : \mathrm{A}$ pure

- No Boolean, no measurement
- Unitary + linear lambda + qif

$\Gamma \vdash_{\text{CPM}} \mathrm{M} : \mathrm{A}$ All CPM

- Every types (classical $\Rightarrow$ copyable)
- meas, $|0\rangle$, …
- No qif

Typing Rule 1: qif

Typing rule

$$\frac{\Gamma \vdash_{\mathrm{Hilb}} N_1 : A \qquad \Gamma \vdash_{\mathrm{Hilb}} N_2 : A}{\Gamma \vdash_{\mathrm{Hilb}} \mathrm{qif}\ M\ \mathrm{then}\ N_1\ \mathrm{else}\ N_2 : \mathrm{qbit} \otimes A} \quad (A\text{: first order})$$

E.g.

$$\frac{\begin{array}{c} y: \mathrm{qbit}, f, g : \mathrm{qbit} \multimap \mathrm{qbit} \vdash_{\mathrm{Hilb}} (g \circ f)\, y : \mathrm{qbit} \\ y: \mathrm{qbit}, f, g : \mathrm{qbit} \multimap \mathrm{qbit} \vdash_{\mathrm{Hilb}} (f \circ g)\, y : \mathrm{qbit} \end{array}}{\begin{array}{l} x, y: \mathrm{qbit}, f, g : \mathrm{qbit} \multimap \mathrm{qbit} \\ \quad \vdash_{\mathrm{Hilb}} \mathrm{qif}\ x\ \mathrm{then}\ (g \circ f)\, y\ \mathrm{else}\ (f \circ g)\, y : \mathrm{qbit} \otimes \mathrm{qbit} \end{array}}$$

Typing Rule 2: Embedding

Typing rule

$$\frac{\Gamma \vdash_{\text{Hilb}} M : A}{\Gamma \vdash_{\text{CPM}} M : A}$$

E.g.

$$\frac{x, y : \text{qbit}, f, g : \text{qbit} \multimap \text{qbit} \vdash_{\text{Hilb}} \text{SWITCH}(f, g, x, y) : \text{qbit} \otimes \text{qbit}}{x, y : \text{qbit}, f, g : \text{qbit} \multimap \text{qbit} \vdash_{\text{CPM}} \text{SWITCH}(f, g, x, y) : \text{qbit} \otimes \text{qbit}}$$

$$\frac{x, y : \text{qbit}, f, g : \text{qbit} \multimap \text{qbit} \vdash_{CPM} \text{SWITCH}(f, g, x, y) : \text{qbit} \otimes \text{qbit}}{x, y : \text{qbit} \vdash_{CPM} \text{let } f = \mathscr{F} \text{ in let } g = \mathscr{G} \text{ in qif } x \text{ then } (g \circ f)\, y \text{ else } (f \circ g)\, y : \text{qbit} \otimes \text{qbit}}$$

Note: f and g cannot be substituted

✗ **qif x then** $(\mathscr{F} \circ \mathscr{E})$ **y**
else $(\mathscr{E} \circ \mathscr{F})$ **y**

Categorical Semantics: Hilb and CPM

Compact closed categories

(F)Hilb: (finite-demensional) Hilbert spaces
+ linear maps

CPM: (biproduct completion of) Hilbert spaces
+ completely positive maps

Strong monoidal functor

$$\iota\colon\ \mathrm{Hilb} \longrightarrow \mathrm{CPM}; \qquad f \longmapsto f(-)f^{\dagger}$$

Categorical Semantics

$\Gamma \vdash_{\text{Hilb}} M : A$

$$[\![M]\!] \in \mathrm{Hilb}([\![\Gamma]\!], [\![A]\!])$$

- No Boolean
- Unitary + linear lambda + qif

$$\downarrow \iota$$

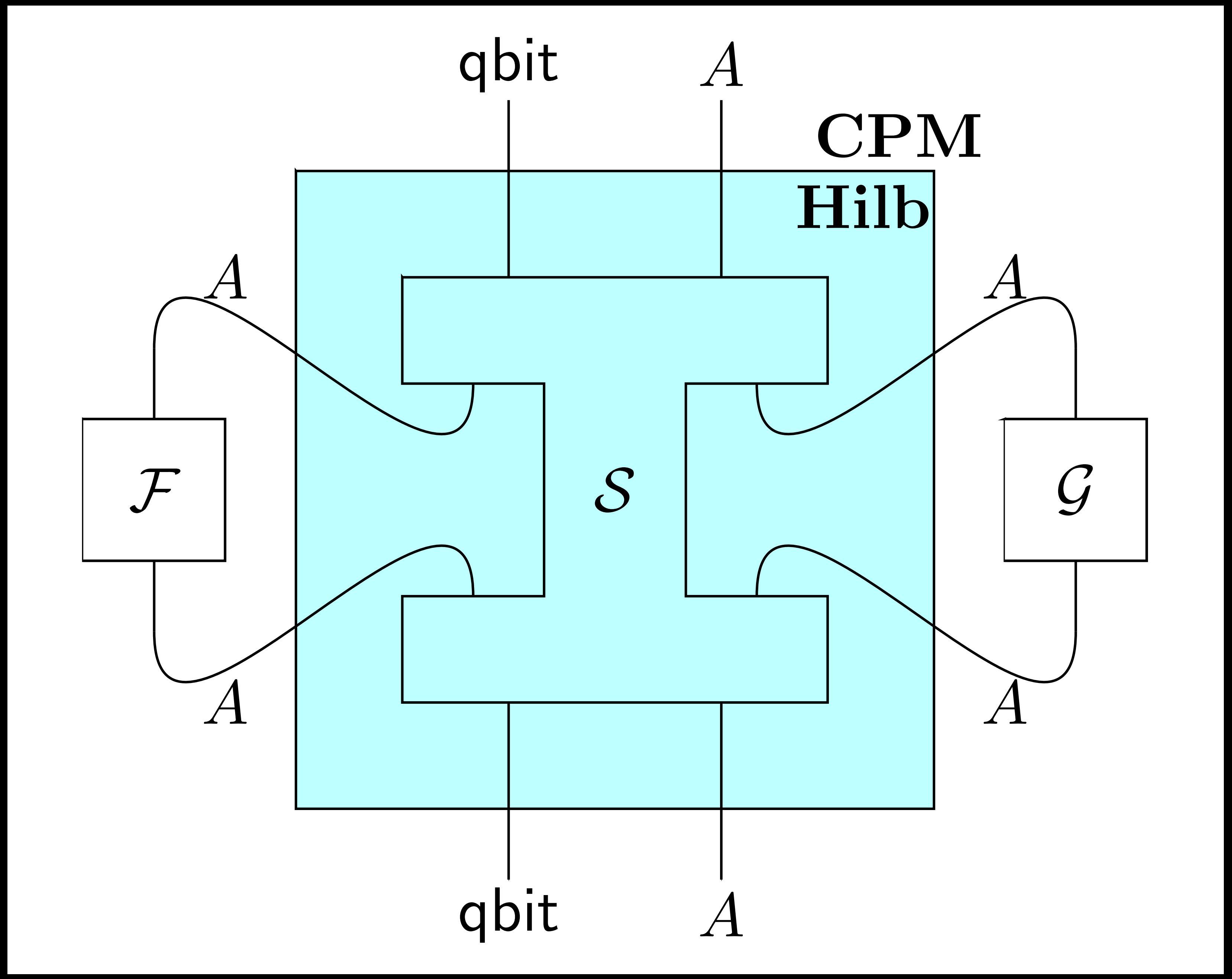
$\Gamma \vdash_{\text{CPM}} M : A$

$$[\![M]\!] \in \mathrm{CPM}([\![\Gamma]\!], [\![A]\!])$$

- Every types (classical $\Rightarrow$ copyable)
- meas, $|0\rangle$, …
- No qif

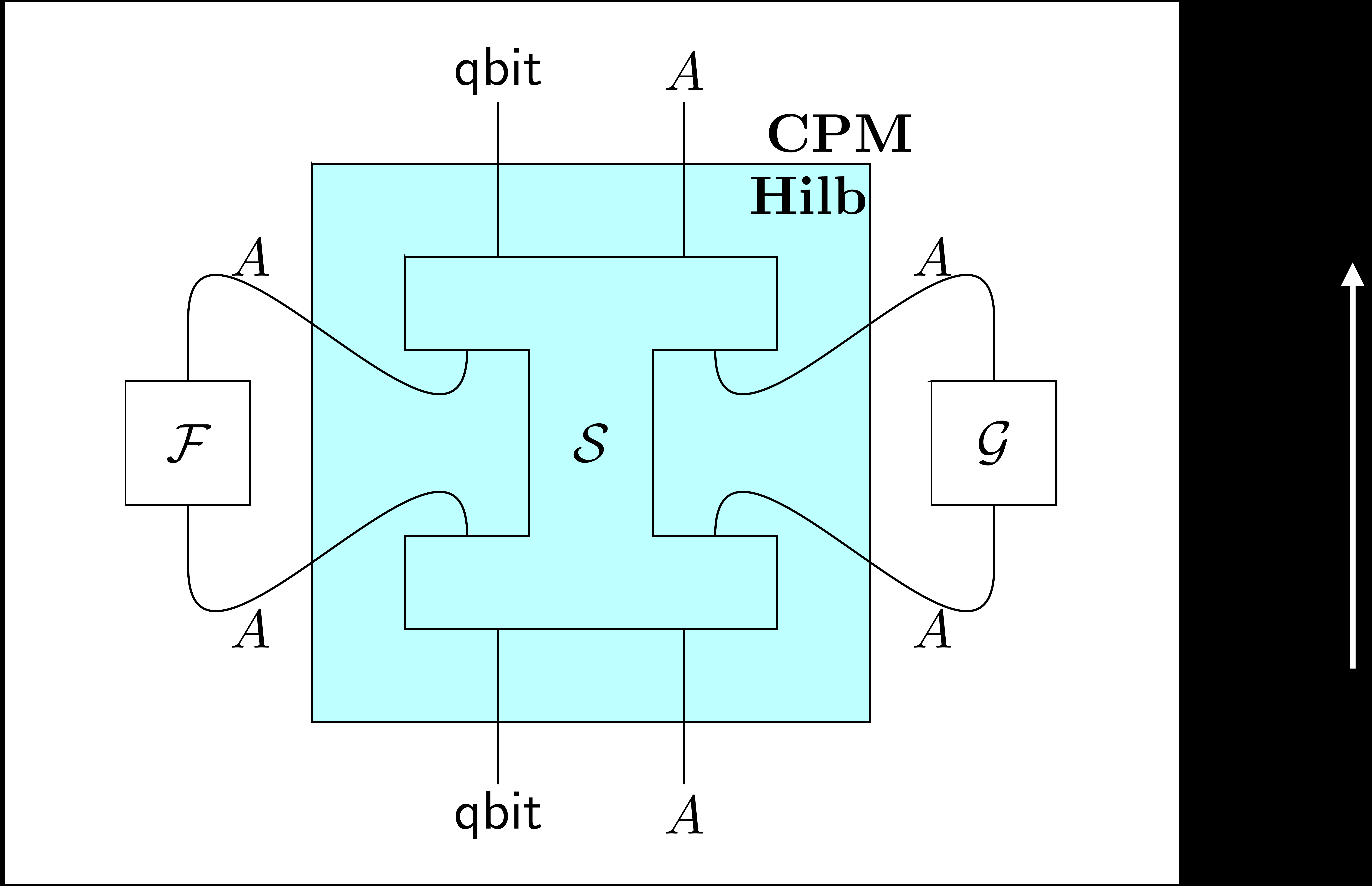
Operational Semantics

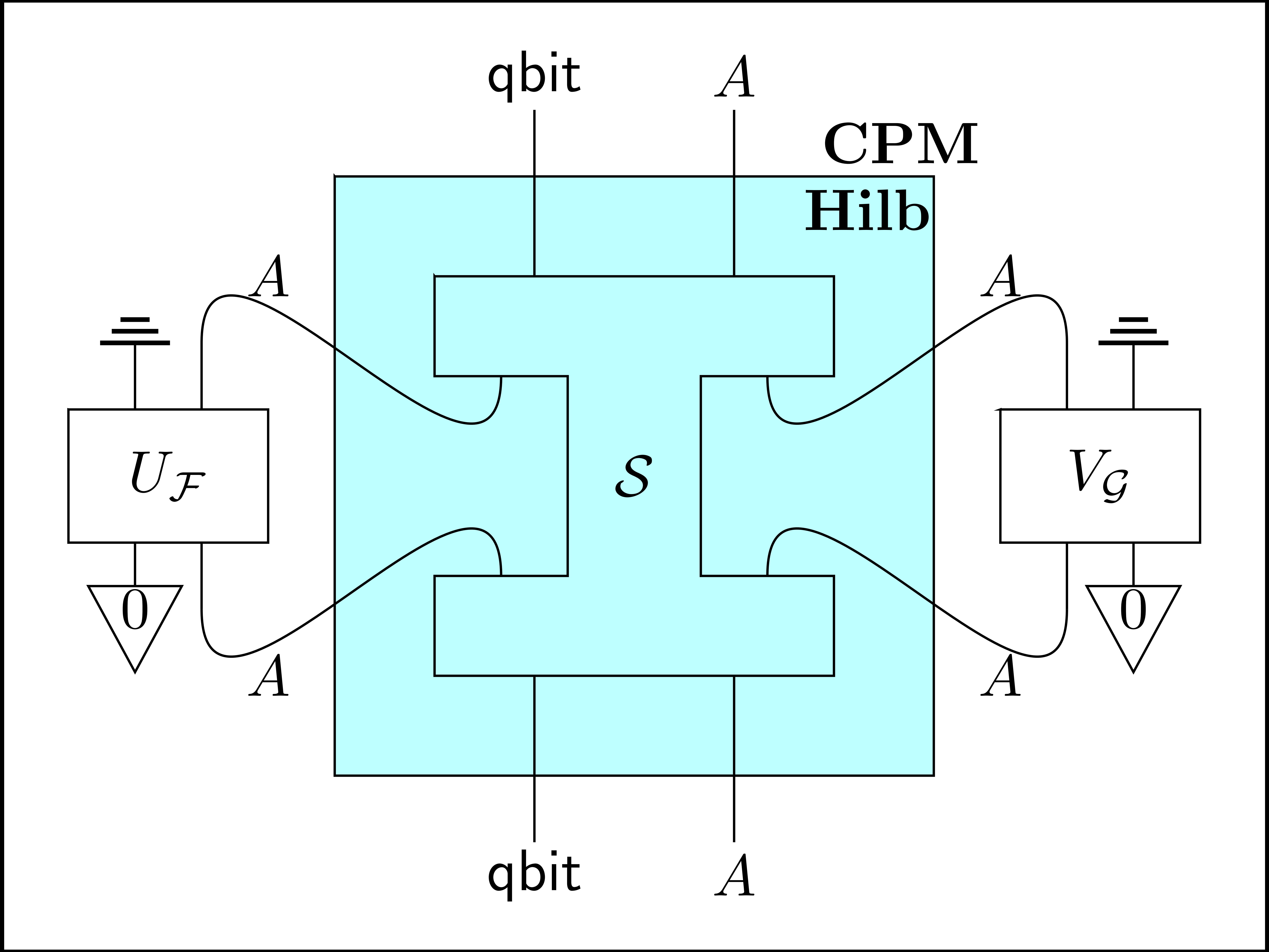
$x, y\text{: qbit} \vdash_{\text{CPM}}$ let f = $\mathscr{F}$ in
let g = $\mathscr{G}$ in
qif x then $(g \circ f)$ y else $(f \circ g)$ y
: qbit $\otimes$ qbit

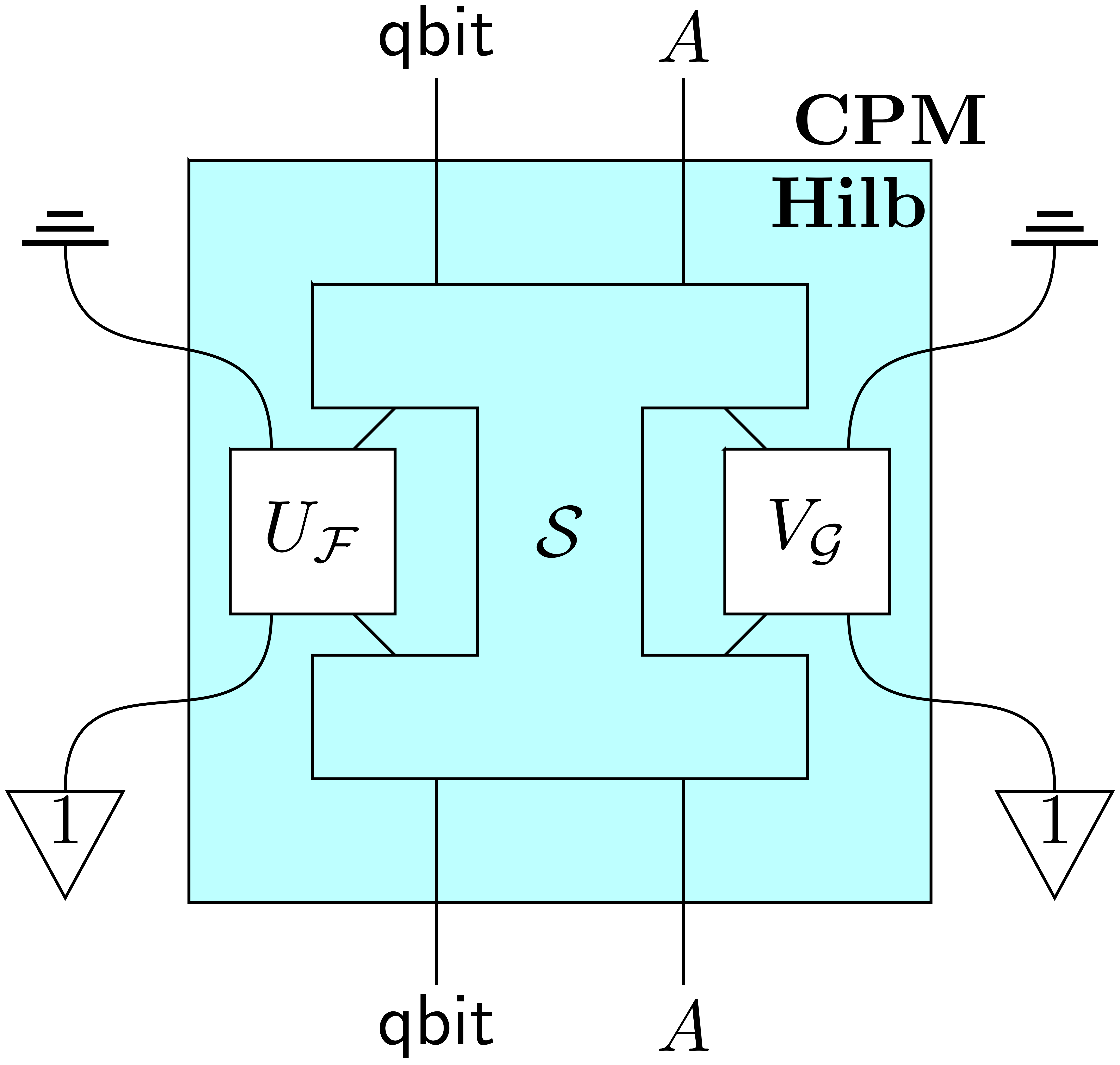


Note: f and g cannot be substituted

✗ qif x then $(\mathscr{F} \circ \mathscr{E})$ y
else $(\mathscr{E} \circ \mathscr{F})$ y


qbit
A
CPM
Hilb
A
A
$\mathcal{F}$
$\mathcal{S}$
$\mathcal{G}$
A
A
qbit
A


qbit
A
CPM
Hilb
A
A
$U_{\mathcal{F}}$
$\mathcal{S}$
$V_{\mathcal{G}}$
0
0
A
A
qbit
A



qbit
A
CPM
Hilb
$U_{\mathcal{F}}$
$\mathcal{S}$
$V_{\mathcal{G}}$
1
1
qbit
A

Summary

- General controlled-$\mathscr{E}$ vs. quantum SWITCH

  ill-defined well-defined

- Observation: Correspondence Problem
- Proposal: Linearity and 2-layer language
  - Canonical categorical semantics (= no choice)
  - Operational semantics by dilation